Readers' Forum

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Comments on Nonlinear Vibrations of Immovably Supported Beams by Finite-Element Method

90006

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SEVERAL recent papers[1-3] have examined the nonlinear vibrations of immovably supported beams using a finite-element formulation that describes the deformation of the beam in terms of a single displacement term—the transverse displacement w of the neutral axis. In fact, a significant amount of work has been done by Mei, Rao, I. S. Raju, K. K. Raju, and Szilard, as reported in both the *AIAA Journal* and the *Journal of Computers and Structures*, in which the large deformation problem of beams and plates has been approached by eliminating the longitudinal degrees-of-freedom. While the error in this approach is easily understood in the case of plates, for beams, it appeared feasible because the corresponding continuum formulation could be done by using the transverse degree-of-freedom alone. This Comment aims at demonstrating that using a Ritz-type finite-element formulation that eliminates the longitudinal degree of freedom may not be possible for the case of beams either.

The class of problems studied here belongs to the moderately large bending theory, where the nonlinearity is due to the stretching of the middle surface or neutral axis due to the immovability of the supports, taking into account moderately large deformation. Such a class of problems is defined by the following conditions:

Elastic strain energy

$$U=\tfrac{1}{2}\int_0^L EI\bar{w}^2{}_{,\bar{x}\bar{x}}\,\mathrm{d}\bar{x}+\tfrac{1}{2}\int_0^L EA\,(\bar{u}_{,\bar{x}}+\tfrac{1}{2}\bar{w}^2{}_{,\bar{x}})^2\,\mathrm{d}\bar{x} \tag{1}$$

Kinetic energy

$$T=\tfrac{1}{2}\int_0^L m\bar{w}_{,t}^2\,\mathrm{d}\bar{x} \tag{2}$$

and the boundary conditions

$$\bar{u}(0)=0,\quad \bar{u}(L)=0 \tag{3}$$

where $\bar{u}$ and $\bar{w}$ represent the longitudinal and transverse deflections of a point on the neutral axis of a beam of length L and mass per unit length m. The boundary conditions on $\bar{w}$ may be

$$\bar{w}(0)\ \text{or}\ \bar{w}_{,\bar{x}}(0)=0$$

and

$$\bar{w}(L)\ \text{or}\ \bar{w}_{,\bar{x}}(L)=0$$

A variational derivation of the differential equations of motion using Lagrange's equations yields the in-plane equation

$$\frac{\mathrm{d}}{\mathrm{d}\bar{x}}\,(\bar{u}_{,\bar{x}}+\tfrac{1}{2}\bar{w}_{,\bar{x}}^2)=0$$

or

$$N=EA\,(\bar{u}_{,\bar{x}}+\tfrac{1}{2}\bar{w}_{,\bar{x}}^2)=\text{const} \tag{4}$$

where N, the axial force at any station, is a constant along the length of the beam.

From Eqs. (3) and (4), it can be shown that

$$N=EA\,(\bar{u}_{,\bar{x}}+\tfrac{1}{2}\bar{w}_{,\bar{x}}^2)=\frac{EA}{L}\int_0^L \tfrac{1}{2}\bar{w}_{,\bar{x}}^2\,\mathrm{d}\bar{x}=\text{const} \tag{5}$$

and Eq. (1) may be replaced by

$$U=\tfrac{1}{2}\int_0^L EI\bar{w}_{,\bar{x}\bar{x}}^2\,\mathrm{d}\bar{x}+\frac{EA}{2L}\left(\int_0^L \tfrac{1}{2}\bar{w}_{,\bar{x}}^2\,\mathrm{d}\bar{x}\right)\left(\int_0^L \tfrac{1}{2}\bar{w}_{,\bar{x}}^2\,\mathrm{d}\bar{x}\right) \tag{6}$$

Equations (2) and (6) suggest that it may be worthwhile to formulate this particular class of problems in a finite element form using an element which allows only the transverse deflection w and its derivatives. This appears to be the rationale behind Refs. 1 and 3, whereas Ref. 2 assumes incorrectly that the stretching energy due to $\bar{u}_{,\bar{x}}$ can be ignored in Eq. (1) itself, in comparison with that due to $\tfrac{1}{2}\bar{w}_{,\bar{x}}^2$.

However, in Refs. 1 and 3, the stiffness matrix of an element of length l is developed by writing the element strain energy in terms of an element displacement function $w(x)$ as

$$U_e=\tfrac{1}{2}\int_0^l EIw_{,xx}^2\,\mathrm{d}x+\frac{EA}{2l}\left(\int_0^l \tfrac{1}{2}w_{,x}^2\,\mathrm{d}x\right)\left(\int_0^l w_{,x}^2\,\mathrm{d}x\right) \tag{7}$$

and this is equivalent to the assumption of Raju et al.[2] of dropping u altogether from Eq. (1). Such a mathematical formulation represents an unreal physical model which allows each point of the undeformed neutral axis to move only vertically downward (i.e., $u=0$ everywhere), even when undergoing large deformation, whereas the elastomechanics of the problem require a description involving both u and w.

Thus, Mei's model results in an axial force N_e which varies from element to element as

$$N_e=\frac{EA}{l}\int_0^l \tfrac{1}{2}w_{,x}^2\,\mathrm{d}x$$

whereas the correct nature of the problem requires that the axial force N depend on the total system deformation. This was pointed out by the first author in Ref. 4 and it was presumed then that a simple correction for N alone would yield correct results.[5] However, subsequent computational work by the present authors using the matrices derived by Mei[1,3] but redefining the axial force N as

$$N=\sum_{m=1}^{M}\frac{EA}{Ml}\left(\int^l \tfrac{1}{2}w_{,x}^2\,\mathrm{d}x\right)=\sum_{m=1}^{M}\frac{N_e}{M}$$

where M is the number of elements, did not yield results consistent with continuum theory. The curx of the problem is that a simple modification as suggested in Ref. 4 is insufficient, since Mei's matrices were derived from a strain energy term given in Eq. (7). It is important to note here that

$$N=\frac{EA}{l}\int_0^l (u_{,x}+\tfrac{1}{2}w_{,x}^2)\,\mathrm{d}x\neq\frac{EA}{l}\int_0^l \tfrac{1}{2}w_{,x}^2\,\mathrm{d}x=N_e$$

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except when $u=0$ at the ends of the element.

The correct strain energy term from which the element stiffness matrix should have been derived would therefore be

$$U_e = \tfrac{1}{2}\int_0^l EIw_{,xx}^2\,dx + \frac{EA}{2L}\left(\int_0^L \tfrac{1}{2}\bar{w}_{,\bar{x}}^2\,d\bar{x}\right)\left(\int_0^l \tfrac{1}{2}w_{,x}^2\,dx\right) \qquad (8)$$

This raises some subtle philosophical problems. In the finite element method, a structure is discretized using a number of finite elements each having structural properties based on displacement assumptions over the element only and independent of structural properties of the rest of the structure. The entire structural system is then built up by assembling such elements and imposing compatibility of displacements at the inter-element boundaries. However, in the present method, the functional to be minimized, $(T_e - U_e)$ as indicated by Eq. (8) contains both element displacement quantities $w_{,x}$ over $x=0$ to l and system displacement quantities $\bar{w}_{,\bar{x}}$ over $\bar{x}=0$ to L. This therefore violates the basic philosophy of the finite element method.

The conclusion to be derived from this exercise is that it may not be possible to develop a Ritz-type finite element analog for the moderately large bending theory of a beam in terms of the transverse displacement function alone, although it is possible in a continuum approach.

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80006

J80-139

Reply by Author to G. Prathap and G. R. Bhashyam

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THE author wishes to thank Dr. Prathap for his interest in the work and for his many comments.[1-5] However, the author does not agree with the conclusion reached in the present Comment. First of all, to find out whether an assumption or a physical model is vaid or not, one should compare results from the model with experiments or other reliable solutions. That was exactly what Prathap and Bhashyam have done by redefining the axial force as an average of element axial forces and comparing the results with continuum solutions. Second, the axial force developed due to large amplitudes, which is treated as constant along the length of the beam, is also an approximation. It is based on an assumption that the longitudinal inertia effects can be neglected. The finite element method[6-8] involves an approximation of the axial force N_e which varies from element to element or an assumption that the axial displacement can be neglected for moderately large amplitude vibrations of a beam with immovable edges. Is this a good and valid assumption? The following clearly gives a positive answer.

1) Figure 3 of Ref. 6 gives a comparison of resonant frequency between the finite element method, three continuum solutions, and experimental data.[9] The finite element approach gives the prediction that agrees best with the experiment.

2) An analytical investigation[10,11] has shown that the effect on nonlinear frequency of inclusion of longitudinal deformation in the strain-displacement relations is not of much significance for moderately large amplitude vibrations of slender beams and thin plates.

3) Furthermore, the finite element results,[6-8,12,13] including the frequency ratios for higher modes (Table 1 of Ref. 6), all were in good agreement with those available in the literature.

In conclusion, the author would like to make a few comments. Since Refs. 6-8 and 12 are the first general formulation by the finite element method for studying the nonlinear vibrational characteristics of beams and thin plates, further modifications are expected to improve upon it. In view of the lack of test data in the literature for large amplitude vibrations, more experimental data are urgently needed.

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